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Stabilization by Multiscale Decomposition

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Abstract—We take advantage of the results of norm characterization by multiscale decomposition to introduce a stabilized formulation of an abstract noncoercive problem. The resulting stabilized problem is positive definite and can therefore be safely discretized by the Galerkin method. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Stabilization, Noncoercive problems, Discretization, Multiscale decomposition.

1. INTRODUCTION

In many problems of mathematical physics, a partial differential equation arises that takes the following form: find $u \in V$, (V a Hilbert space), such that for all $v \in V$, we have $a(u,v) = \langle f, v\rangle$ with f a given element of V' and a a positive *semidefinite* bilinear form. For instance, this is the case of the Stokes equation, and of the equations arising from the application of the Lagrange multiplier method for appending boundary conditions to an elliptic equation. It is well known that when discretizing equations of such a form, instability problems can arise. The remedy is either to choose the discretization space carefully, or to utilize some stabilization technique [1], allowing one to transform the unstable discrete problem into a stable one by adding either suitable elements to the discretization space, or (and this is the case that we are going to consider), suitable consistent terms to the equation itself. In [2], an abstract stabilization technique has been introduced that makes use of the scalar product of the dual space V'. Due to the fact that the computation of the scalar product of a dual space is in general quite cumbersome and not at all easy to handle, Baiocchi and Brezzi [2] introduce such a technique essentially as a way of suggesting more practical stabilization methods, which are obtained by substituting, case by case, the scalar product in the dual space by a scalar product in a more regular space, normalized by a suitable mesh dependent weight. However, the recent results of norm characterization through multiscale decomposition [3] make an equivalent scalar product in dual spaces available for a wide range of Hilbert spaces, including Sobolev spaces. It turns out that when we restrict ourselves to discrete subspaces satisfying a suitable inverse (or *Bernstein*) inequality, such an equivalent scalar product is practically implementable in a finite number of operations (see Lemma 3.1), and the idea of Baiocchi and Brezzi [2] becomes suitable to a practical use. In the following, we will briefly recall the results on norm characterization through multiscale decomposition, and then introduce the new stabilization technique that takes advantage of such results. The result of this paper and its applications are more deeply discussed in [4].

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2. MULTISCALE DECOMPOSITION AND SCALAR PRODUCTS

In this section, we recall the results of Dahmen [3]. For a wide class of Hilbert spaces, it is possible to write down equivalent norms, and equivalent scalar products, by means of a so-called *multiscale decomposition*. Such spaces are essentially those that can be characterized with the aid of a *modulus*.

More precisely, let H be a Hilbert space whose scalar product is denoted by $(\cdot,\cdot)_H$, and let ω be a *modulus* on H, i.e., an application $\omega : H \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following properties hold:

$$\omega(v,t) \leq C\|v\|_H, \tag{1}$$

$$\lim_{t\to 0^+} \omega(v,t) = 0, \tag{2}$$

$$\omega(u+v,t) \leq \omega(u,t) + \omega(v,t). \tag{3}$$

(In the following, we will indicate by C different positive constants, depending in general on the Hilbert spaces and on the multiscale decomposition considered, but independent of the discretization parameters (h) or of the truncation level (J).)

We will consider in the following a continuous scale of Hilbert spaces H^s_ω, defined as follows. Given $\rho > 1$ and $s > 0$, H^s_ω is the subspace of H for which the norm

$$\|v\|_{H^s_\omega} := \|v\|_H + \left(\sum_{j=0}^{+\infty} \rho^{2js}\omega(v,\rho^{-j})^2\right)^{1/2} \tag{4}$$

is finite. For $s < 0$, we will write $H^s_\omega := (H^{-s}_\omega)'$.

Let $\{S_j\}$ be a dense nested sequence of finite-dimensional subspaces of H, and let $Q_j : H \longrightarrow S_j$ be an associated sequence of uniformly bounded linear projectors satisfying

$$Q_j Q_l = Q_l, \qquad \text{for all } l \leq j. \tag{5}$$

Let Q'_j be the adjoint of Q_j, let $\widetilde{S}_j$ be the range of Q'_j, and let the following *Jackson* and *Bernstein* inequalities hold for given s and s', $s, s' > 0$, for the sequences (S_j, Q_j) and $(\widetilde{S}_j, Q'_j)$, respectively:

$$\begin{aligned} &\text{for } v \in H^s_\omega, && \|v - Q_j v\|_H \leq C\rho^{-js}\|v\|_{H^s_\omega}, \\ &\text{for } v \in H^{s'}_\omega, && \|v - Q'_j v\|_H \leq C\rho^{-js'}\|v\|_{H^{s'}_\omega}, \end{aligned} \tag{6}$$

$$\begin{aligned} &\text{for } v_j \in S_j, && \|v_j\|_{H^s_\omega} \leq C\rho^{js}\|v_j\|_H, \\ &\text{for } v_j \in \widetilde{S}_j, && \|v_j\|_{H^{s'}_\omega} \leq C\rho^{js'}\|v_j\|_H. \end{aligned} \tag{7}$$

Under such assumptions, Q_j and Q'_j have bounded extensions to $H^{-s'}_\omega$ and H^{-s}_ω, respectively, and the following norm equivalences hold: for $-s' < t < s$, we have [3, Corollary 5.2]:

$$\|v\|_{H^t_\omega} \approx \left(\sum_{j=0}^{+\infty} \rho^{2jt}\|(Q_{j+1} - Q_j)v\|_H^2\right)^2. \tag{8}$$

The use of the projectors Q_j allows us then to write down an equivalent scalar product for the spaces H^t_ω, as the following corollary states.

COROLLARY 2.1. *For $-s' < t < s$, the bilinear form*

$$[u,v]_t := \sum_{j\geq 0} \rho^{2jt}((Q_{j+1} - Q_j)u, (Q_{j+1} - Q_j)v)_H \tag{9}$$

is a scalar product on H^t_ω inducing a norm which is equivalent to the norm $\|\cdot\|_{H^t_\omega}$.

REMARK 2.1. Among the spaces that enter in the class considered above, for which a norm equivalence of the form (8) holds, we can count Sobolev spaces, that is, Corollary 2.1 holds for the cases $H^s_\omega = H^s(\Omega)$ and $H^s_\omega = H^s_0(\Omega)$, with, in both cases, $H = L^2(\Omega)$.

3. STABILIZATION OF AN ABSTRACT PROBLEM

For some $0 \le \tau < \min\{s, s'\}$, let $V = H^\tau_\omega$. We will denote by $\langle \cdot, \cdot \rangle : V' \times V \longrightarrow \mathbb{R}$ the duality relation between V' and V. Consider a problem of the following form. Given a bilinear form $a : V \times V \longrightarrow \mathbb{R}$, and given $f \in V'$, find $u \in V$ such that $\forall\, v \in V$, we have

$$a(u, v) = \langle f, v \rangle. \tag{10}$$

We denote by $A : V \longrightarrow V'$ the linear operator defined by

$$\langle Au, v \rangle = a(u, v), \qquad \forall\, v \in V. \tag{11}$$

We make the following assumptions on the bilinear form a and on the corresponding linear operator A.

- The bilinear form a is continuous positive semidefinite, i.e., we have for all $u, v \in V$,

$$a(u, v) \le M \|u\|_V \|v\|_V, \qquad a(u, u) \ge 0, \tag{12}$$

  ($M > 0$, constant).
- The operator A is an isomorphism between V and V', i.e., there exists a positive constant L such that

$$\frac{1}{M} \|Au\|_{V'} \le \|u\|_V \le L \|Au\|_{V'}, \tag{13}$$

  (M being the constant appearing in (12)).

Let a_s and a_a be, respectively, the symmetric and antisymmetric part of a:

$$a_s(u, v) := \frac{1}{2}(a(u, v) + a(v, u)), \tag{14}$$

$$a_a(u, v) := \frac{1}{2}(a(u, v) - a(v, u)), \tag{15}$$

and let $A_s : V \longrightarrow V'$ and $A_a : V \longrightarrow V'$ be defined, respectively, as

$$\langle A_s u, v \rangle = a_s(u, v), \quad \langle A_a u, v \rangle = a_a(u, v), \qquad \forall\, v \in V. \tag{16}$$

For any $t \in \mathbb{R}$ and $\epsilon > 0$, we set

$$\begin{aligned} A_t &= A_a + t A_s, \\ a_{\epsilon,t}(u, v) &= a(u, v) + \epsilon [Au, A_t v]_{-\tau}, \\ \langle f_{\epsilon,t}, v \rangle &= \langle f, v \rangle + \epsilon [f, A_t v]_{-\tau}. \end{aligned} \tag{17}$$

Baiocchi and Brezzi [2] give, then, the following theorem.

THEOREM 3.1. *For all $t \in \mathbb{R}$, there exists $\epsilon(t) > 0$, such that $\forall\, \epsilon$, $0 < \epsilon < \epsilon(t)$, the bilinear form $a_{\epsilon,t}$ is continuous and coercive.*

We can then introduce the following problem. Find $u \in V$ such that for all $v \in V$, we have

$$a_{\epsilon,t}(u, v) = \langle f_{\epsilon,t}, v \rangle. \tag{18}$$

It is immediate to check that the solution of (10) is also solution of (18). Then, if $\epsilon < \epsilon(t)$, (18) is an equivalent coercive formulation of (10).

REMARK 3.1. In the case $t = 1$ (i.e., $A_t = A$), one can also consider the following least square problem, which, in a way, would correspond to formally taking $\epsilon = +\infty$, in (17). Find $u \in V$, such that for all $v \in V$, we have

$$a_{LS}(u, v) := [Au, Av]_{-\tau} = [f, Av]_{-\tau}. \tag{19}$$

It is immediate to check, thanks to property (13), that the bilinear form $a_{LS} : V \times V \longrightarrow \mathbb{R}$ is symmetric and coercive, and, also here, the unique solution of problem (10) coincides with the unique solution of problem (19). In other words, such a choice gives us a symmetric coercive formulation of our problem. For this approach, see also [5].

Let us now consider a Galerkin discretization of problem (18). Given a finite-dimensional subspace $V_h \subset V$, find $u_h \in V_h$ such that for all $v_h \in V_h$, one has $a_{\epsilon,t}(u_h, v_h) = \langle f_{\epsilon,t}, v_h\rangle$. Due to the coercivity property of the bilinear form $a_{\epsilon,t}$, any discretization space $V_h \subset V$ can be used. However, the practical computation of the bilinear form $a_{\epsilon,t}$ and of the right-hand side $f_{\epsilon,t}$ implies the need of computing an infinite sum. It is therefore necessary to approximate $[\cdot,\cdot]_{-\tau}$ in such a way that the resulting approximated bilinear form is still coercive at least on the discrete space. Under suitable assumptions on the discretization space, this is possible thanks to the following lemma.

LEMMA 3.1. *Assume that the discretization space V_h satisfies $V_h \subset H_\omega^{\tau+\delta}$, ($\delta > 0$), that $u \in H_\omega^{\tau+\delta}$ implies $A_a u \in H_\omega^{-\tau+\delta}$, and that the following inverse inequality holds:*

$$v_h \in V_h \quad \Rightarrow \quad \|A_a v_h\|_{H_\omega^{-\tau+\delta}} \le C h^{-\delta} \|A_a v_h\|_{V'}, \tag{20}$$

then there exists a $J(h)$ such that for all $v_h \in V_h$, we have for another positive constant C

$$\sum_{j=0}^{J(h)} \rho^{-2j\tau} \|A_a v_h\|_H^2 \ge C \|A_a v_h\|_{V'}^2. \tag{21}$$

PROOF. For simplicity, let us denote $\Delta_j = (Q_{j+1} - Q_j)$. We have

$$\rho^{-j\tau} \|\Delta_j(A_a v_h)\|_H = \rho^{-j\delta} \rho^{j(-\tau+\delta)} \|\Delta_j(A_a v_h)\|_H.$$

This implies

$$\sum_{j \ge J}^{+\infty} \rho^{-2j\tau} \|\Delta_j(A_a v_h)\|_H^2 \le C \rho^{-2J\delta} \|A_a v_h\|_{H_\omega^{-\tau+\delta}}^2 \le C \left(\frac{\rho^{-2J}}{h^2}\right)^\delta \|A_a v_h\|_{V'}^2.$$

Now we have

$$\begin{aligned} \sum_{j=0}^{J} \rho^{-2j\tau} \|\Delta_j(A_a v_h)\|_H^2 &= \sum_{j=0}^{\infty} \rho^{-2j\tau} \|\Delta_j(A_a v_h)\|_H^2 - \sum_{j=J+1}^{\infty} \rho^{-2j\tau} \|\Delta_j(A_a v_h)\|_H^2 \\ &\ge \left(C' - C\left(\frac{\rho^{-2J}}{h^2}\right)^\delta\right) \|A_a v_h\|_{V'}. \end{aligned}$$

By choosing J big enough, we get the thesis. ∎

We can then introduce the following "discrete V' scalar product": for u and v in V', we set

$$[u, v]_{-\tau,h} := \sum_{j=0}^{J(h)} \rho^{-2j\tau} ((Q_{j+1} - Q_j)u, (Q_{j+1} - Q_j)v)_H. \tag{22}$$

We can now introduce the following discrete problem: find u_h in V_h such that for all $v_h \in V_h$, we have:

$$a(u_h, v_h) + \epsilon[Au_h, A_t v_h]_{-\tau,h} = \langle f, v_h\rangle + \epsilon[f, A_t v_h]_{-\tau,h}. \tag{23}$$

The following theorem holds.

THEOREM 3.2. *If V_h satisfies the assumptions of Proposition 3.1, then for all $t \in \mathbb{R}$, there exists an $\epsilon_1(t)$ (independent of h, and depending on the constant C appearing in (20)), such that for all ϵ, $0 < \epsilon < \epsilon_1(t)$, the bilinear form*

$$a_{\epsilon,t,h}(u,v) := a(u,v) + \epsilon[Au, A_t v]_{-\tau,h} \tag{24}$$

is continuous on V and coercive over V_h uniformly in h with respect to the norm $\|\cdot\|_V$, i.e., there exists a constant $\beta(\epsilon) > 0$ independent of h such that we have for all $v_h \in V_h$,

$$a_{\epsilon,t,h}(v_h, v_h) \geq \beta(\epsilon)\|v_h\|_V^2. \tag{25}$$

Moreover, for u and u_h solutions of (10) and (23), respectively, the following error estimate holds:

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{26}$$

PROOF. It is trivial to check that for all $u, v \in V'$, we have: $[u,v]_{-\tau,h} \leq C\|u\|_{V'}\|v\|_{V'}$, from which continuity of $a_{\epsilon,t,h}$ on V easily follows. Let us consider coercivity. We recall that under assumption (12), we have [2]:

$$\|A_s u\|_{V'}^2 \leq Ma(u,u), \tag{27}$$

$$\|Au\|_{V'} \leq \|A_a u\|_{V'} + \sqrt{Ma(u,u)}. \tag{28}$$

Now we have:

$$\begin{aligned}
[Au_h, A_t u_h]_{-\tau,h} &= [A_a u_h, A_a u_h]_{-\tau,h} + (1+t)[A_a u_h, A_s u_h]_{-\tau,h} + t[A_s u_h, A_s u_h]_{-\tau,h} \\
&\geq C_1\|A_a u_h\|_{V'}^2 - (1+|t|)C_2\|A_a u_h\|_{V'}\|A_s u_h\|_{V'} - |t|\|A_s u_h\|_{V'}^2 \\
&\geq \frac{\|A_a u_h\|_{V'}^2}{2} - C_t\|A_s u_h\|_{V'}^2.
\end{aligned}$$

Then, by using (27) we have, for all u_h in V_h,

$$a_{\epsilon,t,h}(u_h, u_h) \geq (1 - \epsilon M C_t)a(u_h, u_h) + \frac{\epsilon}{2}\|A_a u_h\|_{V'}^2, \tag{29}$$

which, thanks to (28) for ϵ small enough, implies the coercivity of $a_{\epsilon,t,h}$.

Now it is easy to check that u solution of (10) and u_h solution of (23) satisfy for all $v_h \in V_h$

$$a_{\epsilon,t,h}(u, v_h) = \langle f, v_h\rangle + \epsilon[f, A_t v_h]_{\tau,h} = a_{\epsilon,t,h}(u_h, v_h). \tag{30}$$

The proof of the error estimate is then easy: for v_h in V_h arbitrary, we have

$$\begin{aligned}
\beta(\epsilon)\|v_h - u_h\|_V^2 &\leq a_{\epsilon,t,h}(v_h - u_h, v_h - u_h) \\
&\leq a_{\epsilon,t,h}(v_h - u, v_h - u_h) + a_{\epsilon,t,h}(u - u_h, v_h - u_h) \\
&\leq M\|u - v_h\|\|v_h - u_h\|.
\end{aligned}$$

By dividing both sides of the inequality by $\|u - u_h\|_V$, we get $\|u_h - v_h\|_V \leq C\|u - v_h\|$. By using the triangular inequality $\|u - u_h\|_V \leq \|u - v_h\|_V + \|v_h - u_h\|_V$, since v_h is arbitrary, we get the thesis. ∎

REMARK 3.2. One of the ways of realizing a multiscale decomposition with the required properties is through the use of *biorthogonal wavelet bases*. In such a case, the results of Lemma 3.1 can be sharpened thanks to the localization properties of such bases. The global $J(h)$ individuated by such lemma can be substituted by a local $J(h)$, depending on the local mesh size.

REMARK 3.3. The same stabilization technique also applies for problems defined on spaces of the form $V = H_\omega^{\tau_1} \times \cdots \times H_\omega^{\tau_n}$. In fact, also for these spaces, it is possible to write down an equivalent scalar product on the dual space. The result of Theorem 3.1 holds unchanged, as do, with the obvious modifications, the results of Lemma 3.1 and of Theorem 3.2. In particular, this technique applies to the case of saddle point problems [6,7].

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